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Validity of scaling relations in absorbing phase transitions with a conserved field

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The two scaling relations in absorbing phase transitions, $\nu_{\parallel}=\beta/\theta$ and $z=\nu_{\parallel}/\nu_{\perp}$, are studied for a conserved lattice gas model. The critical indices calculated elaborately from the all-sample average density of active particles appear to satisfy both relations. However, the exponent $\nu_{\perp}$ calculated from the surviving samples does not appear to be consistent with the value in the thermodynamic limit. This is in contrast with earlier observations [M. Rossi *et al.*, Phys. Rev. Lett. **85**, 1803 (2000); S. Lübeck and P. C. Heger, Phys. Rev. E. **68**, 056102 (2003)], in that the former scaling relation was claimed to be violated.

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The absorbing phase transition (APT) is a prototypical example of nonequilibrium phase transitions and is observed in vast areas of physics, chemistry, and biology [1,2]. The critical behavior of the APT can be categorized into a finite number of universality classes, classified by the number of symmetries of absorbing states and the conservation laws [3–10]. Recently, a different universality class was proposed by Rossi *et al.* [11] for the model with a conserved field. The conserved lattice gas (CLG) model, conserved threshold transfer process (CTTP), stochastic reaction-diffusion models, and sandpile models were found to belong to this universality class [12–15].

In the CLG model, each lattice site may be occupied by at most one particle, and a particle is defined to be active if it has at least one particle in the nearest-neighbor sites; otherwise, it is inactive. The dynamics proceeds with the hopping of active particles; each active particle attempts to hop to one of the neighboring empty sites. During the process, the density of active particles, ρ_a, decreases in time and exhibits the power-law behavior $\rho_a(t) \propto t^{-\theta}$ at critical density ρ_c. For $\rho > \rho_c$, ρ_a converges to the steady-state value $\rho_{\rm sat}$, which exhibits the power-law behavior against the distance from criticality, i.e., $\rho_{\rm sat}(\rho) \propto (\rho-\rho_c)^{\beta}$ for $\rho > \rho_c$. The off-critical values of ρ_a depend on the evolution time and the distance from criticality via the correlation time $\tau \sim |\rho-\rho_c|^{-\nu_{\parallel}}$. Thus, $\rho_a(t)$ can be written as

$$\rho_a(t) = t^{-\theta}\mathcal{F}(t/\tau) = t^{-\theta}\mathcal{F}(t|\rho-\rho_c|^{\nu_{\parallel}}), \tag{1}$$

where $\mathcal{F}(x)$ is the universal off-critical scaling function. Since $\rho_a \to \rho_{\rm sat}$ in the $t \gg \tau$ limit, the scaling relation

$$\beta = \nu_{\parallel}\theta \tag{2}$$

follows. For a finite system, since the correlation length cannot exceed the size of the system near the critical point, i.e., $\xi \sim |\rho-\rho_c|^{-\nu_{\perp}} \sim L$, it is obtained that $|\rho-\rho_c| \sim L^{-1/\nu_{\perp}}$. Therefore, Eq. (1) can be rewritten as

$$\rho_a(t) = t^{-\theta}\mathcal{G}(t/L^z), \tag{3}$$

where $\mathcal{G}(x)$ is the finite-size scaling function and

$$z = \nu_{\parallel}/\nu_{\perp} \tag{4}$$

is the dynamic exponent.

Rossi *et al.* claimed that the scaling relation in Eq. (2) broke the "simple scaling." Failure of the scaling was reported in the APT with a conserved field by Vespignani *et al.* [16] and was also discussed in other works [11,15,17]. The cause of the failure was conjectured to be the anomalous exponent θ. Lübeck and Heger claimed that $\nu_{\parallel}$ obtained from Eq. (2) using the estimates of θ and β was incorrect, and they calculated $\nu_{\parallel}$ from Eq. (4) using the values of $\nu_{\perp}$ and z estimated from finite-size scalings of $\rho_{\rm sat}$ [17]. Lübeck and Misra calculated $\nu_{\parallel}$ from the persistence distribution and obtained a consistent result [18].

Recently, various critical exponents for the CLG model in one dimension were calculated by the present authors. It was found that Eq. (2) held with the measured exponents, whereas Eq. (4) did not hold [19]. In this Rapid Communication, motivated from the work in one dimension, the two scaling relations are carefully examined for the CLG model on a square lattice. It is found that, for finite-size systems, the exponent $\nu_{\perp}$ for the all-sample average appears to be different from that for the surviving-sample average. Accepting that the former is the value valid in the thermodynamic limit, both scaling relations appear to hold.

Simulations are carried out for the CLG model on a square lattice of size L with periodic boundaries using the sequential update rule. Initially, ρL^2 particles are distributed randomly in a system and, at each time step, an active particle is selected and hops to one of its nearest-neighbor empty sites, with an increment of the evolution time $\Delta t = 1/N_a$, N_a being the number of active particles. Determining the critical density is not simple since, for a density close to ρ_c, ρ_a displays a power-law behavior for several decades and, afterwards, it decreases rapidly due to the finite-size effect, as was found in earlier works [11,20]. If ρ_c is determined from the power law of ρ_a in the long-time limit, it will be overestimated and, with the estimate, the power-law behavior of $\rho_{\rm sat}$ against $\rho-\rho_c$ and both the off-critical scaling, and the finite-size scaling will not be satisfactory. In this work, ρ_c is predetermined from the power-law behavior of ρ_a and, with this value, the power law of $\rho_{\rm sat}$, the off-critical scaling and the finite-size scaling are analyzed. The value of ρ_a

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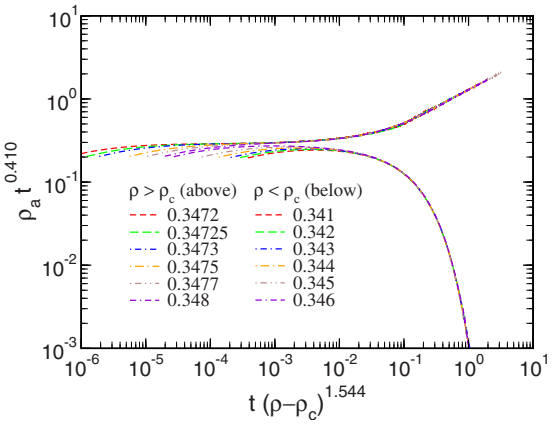

FIG. 1. (Color online) The off-critical scaling function $\rho_a t^\theta$ plotted against $t|\rho-\rho_c|^{\nu_\parallel}$ for selected densities, using $\theta=0.410$ and $\nu_\parallel=1.544$, for the CLG model on a square lattice.

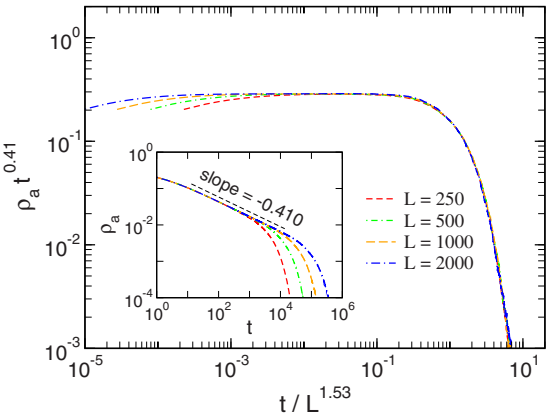

FIG. 2. (Color online) The finite-size scaling function $\rho_a t^\theta$ at ρ_c plotted against t/L^z, using $\theta=0.410$ and $z=1.53$, for the CLG model on a square lattice. The inset is the unscaled data.

which exhibits the best results for these tests is determined as ρ_c.

The critical density is obtained as $\rho_c=0.347\,103$, and $\rho_a(t)$ at ρ_c yields the exponent $\theta=0.410(4)$. The exponent β is calculated from the data for $\rho_{\rm sat}$; the power-law fit against the distance from criticality yields the exponent $\beta=0.633(7)$. The values of θ and β are consistent with those of Rossi *et al.* obtained using a parallel update and also with those of Lübeck using the sequential update; however, the value of ρ_c is larger than that by Lübeck [21]. The difference might be due to the different method of determining ρ_c. Suppose that ρ_c is determined from the best power-law fit of $\rho_{\rm sat}$ against $\rho-\rho_c$ using the samples surviving (remaining in an active phase) up to predetermined time steps. Since for any finite-size system there remain some surviving samples even below criticality, the value of ρ_c estimated in this way with surviving samples would be smaller than the true ρ_c, because the densities at which samples remain in an active phase would be assumed supercritical. Lübeck and Heger obtained ρ_c by this way for the CTTP model [17], and it is suspected that Lübeck employed a similar method for the CLG model as well. In this work, data of $\rho_{\rm sat}$ are used for the densities for which no sample falls into the absorbing phase.

The scaling relation in Eq. (2) is better examined by the off-critical scaling of $\rho_a(t)$ using the estimates of θ and $\nu_\parallel$. It should be noted that the off-critical scaling analysis has not been employed in earlier works for the model with a conserved field [11,16,17], though it is frequently employed to other models in the APT. Plotted in Fig. 1 is the scaled density $\rho_a t^\theta$ against the scaled time t/τ, using $\nu_\parallel=\beta/\theta=1.544$. Data for various densities collapse onto two separate curves, one for $\rho>\rho_c$ (above) and the other for $\rho<\rho_c$ (below), indicating that scaling holds excellently. This confirms that the scaling relation in Eq. (2) is valid.

The finite-size scaling in Eq. (3) is examined with the data averaged over *all samples* at ρ_c for various size systems. Assuming that z is an adjustable parameter for scaling, the best collapse of the data for ρ_a for various size systems is examined. Figure 2 shows the best scaling obtained using $z=1.53$, which is consistent with the estimate by Rossi *et al.*, $z=1.52$ [11]. If the exponent $\nu_\perp$ is calculated using the scaling relation in Eq. (4), $\nu_\perp=\nu_\parallel/z=1.544/1.53\simeq 1.01$ would be obtained. The plots in the inset of Fig. 2 are the unscaled data, which yield $\theta=0.410$.

Focusing on the late time, Eq. (3) is rewritten as

$$\rho_a(t) = L^{-\beta/\nu_\perp}\mathcal{H}(t/L^z), \tag{5}$$

where $z\theta=\beta/\nu_\perp$ has been used. The scaling in Eq. (5) is the simple scaling claimed to be broken by an anomalous exponent θ. This scaling function is particularly useful when ρ_a becomes constant in the region $t \gg L^z$. Since ρ_a for the all-sample averages decays in the long-time limit, as was seen in Fig. 2, those samples which survive up to time t should be employed for this scaling. The exponent $\nu_\perp$ can then be measured from $\rho_a(t)\to\rho_{\rm sat}\propto L^{-\beta/\nu_\perp}$ as $t\to\infty$ at ρ_c. Figure 3 shows ρ_a for selected system sizes. The inset shows the steady-state densities against the system size, with the power $\beta/\nu_\perp=0.792(5)$. Using $\beta=0.633$ obtained earlier, it is obtained that $\nu_\perp=0.799$, which is consistent with the result by Rossi *et al.* [11] and also with that by Lübeck and Heger [17].

Equation (5) implies that the plots of $\rho_a L^{\beta/\nu_\perp}$ against t/L^z should fall on the same curve. However, the scaled data of the surviving samples using $\beta/\nu_\perp$ estimated from the steady-

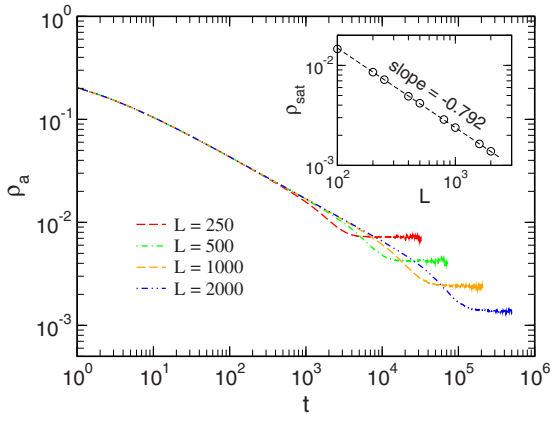

FIG. 3. (Color online) The surviving-sample average of $\rho_a(t)$ against the time at ρ_c for the CLG model on a square lattice. The inset shows the steady-state density against the size of the system.

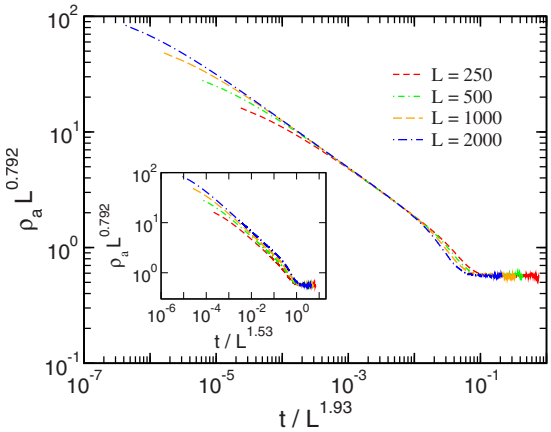

FIG. 4. (Color online) The scaled density of active particles $\rho_a L^{\beta/\nu_\perp}$ for the surviving-sample averages at ρ_c against the scaled time t/L^z, using $\beta/\nu_\perp=0.792$ and $z=1.93$, for the CLG model on a square lattice. Plotted in the inset are the data scaled with $z=1.53$.

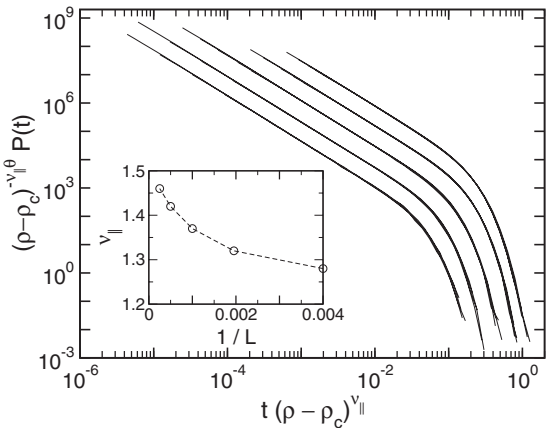

FIG. 5. Plot of the data for $P(t)|\rho-\rho_c|^{-\nu_\parallel\theta_g}$ in a phase of $\rho_a<\langle\rho_a\rangle$ against $t|\rho-\rho_c|^{\nu_\parallel}$. The plots are for, from upper-right to lower-left, $L=250$, 512, 1000, 2000, and 4000, using, respectively, the values $(\theta_g,\nu_\parallel)=(1.64,1.28)$, (1.63, 1.32), (1.62, 1.37), (1.61, 1.42), and (1.615, 1.46). Data are shifted vertically to avoid overlapping. The inset is the estimates of $\nu_\parallel$ against the inverse of the system size.

state density and z from the finite-size scaling do not collapse as shown in the inset of Fig. 4—i.e., simple scaling is broken—as claimed earlier [11,16]. Failure of the scaling may be attributed to the fact that the exponent z obtained from the finite-size scaling of all-sample data was employed to the scaling plot of the surviving sample data. If the scaling in Eq. (5) is examined with the all-sample data, the scaling plot would be that of Fig. 2 multiplied by $x^{-\theta}$, where $x \equiv t/L^z$, i.e., $\mathcal{H}(x)=x^{-\theta}\mathcal{G}(x)$. The scaling, thus, holds with the same quality as in Fig. 2, using $\beta/\nu_\perp=z\theta\simeq 0.63$ rather than that obtained from Fig. 3. Therefore, with $\nu_\perp=\beta/z\theta=1.01$, both relations in Eqs. (2) and (4) are valid, and the off-critical scaling and the finite-size scaling are satisfactory with the all-sample average data.

When the scaling in Eq. (5) is examined with the surviving-sample data using the exponents obtained from the surviving samples, i.e., using $\beta/\nu_\perp=0.792$ and $z=\nu_\parallel/\nu_\perp=1.544/0.799\simeq 1.93$, data for various size systems appear to collapse in two extreme regions $t\ll L^z$ and $t\gg L^z$, as shown in Fig. 4. However, data between the two inflection points do not fall on the same curve. Such inflection points appear for the surviving samples since the time when the finite-size effect comes into the system is different from the time when saturation sets in. These two different time scales might be the cause of the failure of the scaling. For systems without a conserved field, the time when the finite-size effect comes into the system coincides with the time when saturation sets in [22,23]; i.e., there is only one time scale regarding the finite-size effect, $\tau\sim L^z$. However, for systems with a conserved field, there exist two time scales on the surviving-sample data, one $\tau_1\sim L^{z_1}$ with $z_1=1.53$ and the other $\tau_2\sim L^{z_2}$ with $z_2\neq z_1$. ($z_1=1.53$ was verified and z_2 appeared to be about 1.75.) The latter time exists only on the surviving-sample data and appears to have influenced the value of $\nu_\perp$. Since the two time scales are different, data for various size systems do not collapse by a single rescaling of the evolution time. In the thermodynamic limit of $L\to\infty$, the time scales associated with the finite-size effect disappear and a single value of $\nu_\perp$ would be measured. Therefore, the value $\nu_\perp$ which has been influenced by the time scale regarding the finite-size effect will not be compatible with the value in the thermodynamic limit. It should also be noted that averaging over surviving samples could not be the correct way of approaching the thermodynamic limit as long as the finite size is concerned. Thus, $\nu_\perp=1.01$ is the value compatible with that in the thermodynamic limit. Unfortunately, in earlier works, the value of $\nu_\perp$ calculated from the surviving-sample data was employed [11,15,17].

Recently, Lübeck and Misra calculated $\nu_\parallel$ from the persistence distribution and claimed that the obtained value was not consistent with that obtained from Eq. (2) [18]. The persistence distribution $P(t)$—i.e., the distribution of the average time that the system persists in one of the phases, e.g., in the phase that the density of active particles is larger than the mean density $\langle\rho_a\rangle$—is known to scale near the criticality as

$$P(t) = t^{-\theta_g}\mathcal{P}_1(t|\rho-\rho_c|^{\nu_\parallel}) = |\rho-\rho_c|^{\nu_\parallel\theta_g}\mathcal{P}(t|\rho-\rho_c|^{\nu_\parallel}), \quad (6)$$

where θ_g is the global persistence exponent. The exponent $\nu_\parallel$ was estimated to be $\nu_\parallel=1.15$, which was consistent with that obtained from Eq. (4) using the values of $z=1.53$ and $\nu_\perp=0.799$, but was different from the value obtained from Eq. (2) using the estimates of β and θ. In this work, the same simulations are carried out on selected sizes from $L=250$ to as large as $L=4000$, focusing on the dependence of the measurements on the size of the system. The steady-state density $\rho_{\rm sat}$ is first calculated for selected values of ρ in the supercritical region and, after the system goes into the steady state, the distribution of the average time that the system remains in the same phase is calculated. It is found that the exponent $\nu_\parallel$ which yields the optimal scaling for $\rho_a<\langle\rho_a\rangle$ is different from that for $\rho_a>\langle\rho_a\rangle$, and the dominant one is the former. Since it is believed that the dominant one is the true exponent, the scaled data for $\rho_a<\langle\rho_a\rangle$ are presented in Fig. 5. The value of $\nu_\parallel$ which yields the best collapsing is found to depend on L; the measured values are $\nu_\parallel=1.28$, 1.31, 1.37, 1.42, and 1.46 for, respectively, $L=250$, 512, 1000, 2000,

and 4000. Attempting many different ways to extrapolate the results to the $L \to \infty$ limit, it is found that a plot of the estimates against $(\ln L)^{-1}$ yields nearly linear behavior and the intercept on the ordinate is $\nu_{\|}=1.54$, which is consistent with the value obtained from Eq. (2). Therefore, the persistence distribution analysis also supports the validity of Eq. (2).

The dynamic exponent z can also be calculated from the mean spreading distance. Close to the critical point, the correlation length and the correlation time are related to each other as $\xi \sim \tau^{\nu_\perp/\nu_{\|}}$. At ρ_c, ξ cannot exceed the mean spreading distance, i.e., $\xi \sim R$, and τ scales as $\tau \sim t$. Therefore, $R \sim t^{1/z}$, with the scaling relation in Eq. (4). The exponent z may be calculated from the evolution of activity in the close vicinity of absorbing states. From one of the many absorbing states, the system is disturbed by moving a particle to one of the nearest-neighbor sites. Then, the moving particle and its neighboring particles become active. The dynamic simulation, thus, begins from the perturbed state, and the mean spreading distance is calculated against the evolution time. It is obtained that $z=1.54(1)$ (not shown), which is consistent with the value obtained from the finite-size scaling analysis.

The measurement of z is also examined with the data from the free boundaries. The value of ρ_c is obtained to be slightly larger, but the exponents β and θ are consistent with those for the periodic boundaries. The off-critical scaling is held with the estimates, indicating that the boundary condition does not influence the exponent $\nu_{\|}$. However, the finite-size scaling is held with a different value of $z=1.62$. The dynamic simulation using the free boundaries also yields a consistent result of $z=1.63$. This implies that the exponent z and, accordingly, $\nu_\perp$ are vulnerably influenced by the finite-size effect. It is believed that the value obtained using periodic boundaries is more reliable.

In summary, the scaling relations widely known in an APT were investigated for the CLG model. It was found that the known scaling relations were satisfied with the exponents calculated from the all-sample average density of active particles. The off-critical scaling and the finite-size scaling were also satisfied with the estimates. However, the value of $\nu_\perp$ calculated from the surviving samples appeared to be invalid in the thermodynamic limit. The earlier conclusion for the violation of one of the scaling relations, thus, appeared to be attributed to the incorrect exponent $\nu_\perp$. Similar analysis was also performed for the CTTP model, and basically the same conclusion was derived.

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[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, England, 1999).
[2] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[3] J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).
[4] H. Takayasu and A. Y. Tretyakov, Phys. Rev. Lett. **68**, 3060 (1992).
[5] I. Jensen, Phys. Rev. E **50**, 3623 (1994).
[6] S. Kwon and H. Park, Phys. Rev. E **52**, 5955 (1995).
[7] J. Cardy and U. C. Täuber, Phys. Rev. Lett. **77**, 4780 (1996).
[8] G. Ódor, Phys. Rev. E **62**, R3027 (2000).
[9] H. Hinrichsen, Phys. Rev. E **63**, 036102 (2001).
[10] K. Park and I.-M. Kim, Phys. Rev. E **66**, 027106 (2002).
[11] M. Rossi, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **85**, 1803 (2000).
[12] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E **62**, R5875 (2000).
[13] S. S. Manna, J. Phys. A **24**, L363 (1991).
[14] D. Dhar, Physica A **263**, 4 (1999).
[15] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000).
[16] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998); see also R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998).
[17] S. Lübeck and P. C. Heger, Phys. Rev. E **68**, 056102 (2003); S. Lübeck, *ibid.* **66**, 046114 (2002).
[18] S. Lübeck and A. Misra, Eur. Phys. J. B **26**, 75 (2002).
[19] S.-G. Lee and S. B. Lee, Phys. Rev. E **77**, 021113 (2008).
[20] S. B. Lee and Y. N. Kim, Phys. Rev. E **76**, 031137 (2007).
[21] S. Lübeck, Phys. Rev. E **66**, 046114 (2002).
[22] I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993).
[23] W. M. Hwang, S. Kwon, H. Park, and H. Park, Phys. Rev. E **57**, 6438 (1998).